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THE CONVERGENCE OF TWO SERIES REPRESENTATIONS FOR ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST AND SECOND KIND

> James N. Walbert Kathleen L Zimmerman

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February 1980



US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND BALLISTIC RESEARCH LABORATORY ABERDEEN PROVING GROUND, MARYLAND

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| The regions of convergence and divergence of two associated Legendre functions of the first and secon | series representations for and kind, degree v and order u. |
| lare investigated. The series are shown to be absolu | itely convergent when the real |
| hart of the complex order u is less than zero, condi | tionally convergent when the |
| real part of μ is greater than or equal to zero and gent when the real part of μ is greater than or equal | less than one-half, and diver |
| calculations for several real values of v and u are | used to demonstrate the be- |
| havior of the partial sums of the series in the regi | ons of convergence and |

divergence.

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LIST OF SYMBOLS

- θ Apex angle of a cone, in radians
- $\Gamma(x)$ The Gamma function
- $(x)_k = \Gamma(x+k)/\Gamma(x)$, the Pochhammer notation
- $k! = \Gamma(k+1)$, factorial notation
- a,b,c arbitrary complex numbers
- P_{ν}^{μ} Associated Legendre function of the first kind, of degree ν and order $\mu.$
- Q_{ν}^{μ} Associated Legendre function of the second kind, of degree ν and order $\mu.$
- Re {a} the real part of the complex number a.
- Im {a} the imaginary part of the complex number a.
- $\sinh x$, $\cosh x$ the hyperbolic sine and $\cosh x$, respectively, of x.
- $i^2 = -1$
- O(k) Landau symbol

Other terms are defined as they occur in the text.

I. INTRODUCTION

For stresses in a pressurized cone, the method of Papkovich-Neuber leads to displacements and strains expressed in terms of Legendre functions. In spherical coordinates, the argument at which these Legendre functions are evaluated is the cosine of the apex angle θ of the cone. Computation of Legendre functions using Gaussian quadratures on certain integral representations 2 is required in the stress analysis for the minihat gage, a strain-type pressure transducer. 3

For the purpose of verification of the quadrature results, use was made of the series representations

$$P_{\nu}^{\mu} (\cos \theta) = \pi^{-\frac{1}{2}} 2^{\mu+1} (\sin \theta)^{\mu} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}$$

$$\times \sum_{k=0}^{\infty} \frac{(\mu+\frac{1}{2})_{k} (\nu+\mu+1)_{k}}{k! (\nu+\frac{3}{2})_{k}} \sin[(2k+\mu+\nu+1)\theta]$$

$$Q_{\nu}^{\mu} (\cos \theta) = \pi^{\frac{1}{2}} 2^{\mu} (\sin \theta)^{\mu} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}$$

$$\times \sum_{k=0}^{\infty} \frac{(\mu+\frac{1}{2})_{k} (\mu+\nu+1)_{k}}{k! (\nu+\frac{3}{2})_{k}} \cos[(2k+\mu+\nu+1)\theta],$$
(2)

where $0 < \theta < \pi$. In at least two references^{4,5} there

¹A. I. Lure', Three Dimensional Problems of the Theory of Elasticity, 1964, paragraph 1.10.

²W. Magnus, and F. Oberhettinger, <u>Formulas and Theorems for the Special</u> Functions of Mathematical Physics, 1949, p67.

³H. Gay, "The Evolution of Gages for Measuring Pressures in Guns and Pockets at the Ballistic Research Laboratories", BRL Memorandum Report No. 1402, 1962. (AD #283309)

⁴Handbook of Mathematical Functions, National Bureau of Standards, U.S. Department of Commerce, 1964, 8.7.1, 8.7.2.

⁵A. Erdelyi, et al, <u>Higher Transcendental Functions</u>, Vol. I, Bateman Manuscript Project, 1953, 3.5(2), 3.5(3).

are no stated restrictions on μ or ν for the convergence of these series. However, calculation of the first 50 partial sums of the series in equation (1) with $\theta = \frac{\pi}{6}$, $\mu = 1$, and $\nu = .05$ indicated that these partial sums were increasing without bound (see Appendix A).

Indeed, since

$$\frac{(\mu + \frac{1}{2})_{k}(\mu + \nu + 1)_{k}}{k! (\nu + \frac{3}{2})_{k}} = \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2}) \Gamma(\mu + \nu + 1)} \frac{\Gamma(\mu + \frac{1}{2} + k)\Gamma(\mu + \nu + 1 + k)}{\Gamma(1 + k)\Gamma(\nu + \frac{3}{2} + k)},$$
 (3)

one sees at a glance that for real μ and ν with $\mu \geq \frac{1}{2}$, the term in brackets is greater than or equal to 1 for every positive integral k, giving scant hope for the convergence of the series in Eqs. (1) and (2) in this case.

It is the purpose of this report to identify the region of convergence of these series, and to indicate by numerical computation the behavior of the partial sums for various values of the parameters μ and ν .

In reference 4, for example, the convergence criteria for series representations of the Legendre functions are not complete, and should be supplemented by the more general convergence criteria for the hypergeometric function.

II. ANALYSIS OF CONVERGENCE

To simplify our notation, we write

$$a=\mu+\frac{3}{2}$$

 $b=\mu+\nu+1$

$$c = v + \frac{3}{2}$$
. (4)

The series in Eqs. (1) and (2) are then

$$S = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} \sin [(2k+b)\theta] , \qquad (5)$$

and

$$C = \sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{k!(c)_{k}} \cos[(2k+b)\theta] , \qquad (6)$$

respectively.

Since

$$\sin \left[(2k+b)\theta \right] = \frac{1}{2i} \left(e^{i(2k+b)\theta} - e^{-i(2k+b)\theta} \right) ,$$

and

$$\cos [(2k+b)\theta] = \frac{1}{2} \left(e^{i(2k+b)\theta} + e^{-i(2k+b)\theta} \right) ,$$

we can write

$$2iS = \sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{k!(c)_{k}} \left[e^{ib\theta} (e^{i2\theta})^{k} - e^{-ib\theta} (e^{-i2\theta})^{k} \right]$$

$$= e^{ib\theta} \sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{k!(c)_{k}} (e^{i2\theta})^{k} - e^{-ib\theta} \sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{k!(c)_{k}} (e^{-i2\theta})^{k}$$

$$= e^{ib\theta} F(a,b;c;e^{i2\theta}) - e^{-ib\theta} F(a,b;c;e^{-i2\theta});$$

similarly,

$$2C=e^{ib\theta}F(a,b;c;e^{i2\theta})+e^{-ib\theta}F(a,b;c;e^{-i2\theta})$$
,

where

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k$$
 (7)

is the hypergeometric function. These representations for S and C as

linear combinations of hypergeometric functions are valid only on the intersections of the regions of convergence of the various forms in which the series in Eq. (7) appears.

It is known (see Appendix B) that on |z|=1, the series in Eq. (7) exhibits the following behavior:

- I) Absolute convergence if Re{a+b-c}< 0;</p>
- II) Conditional convergence if $z \neq 1$ and $0 < Re\{a+b-c\} < 1$;
- III) Divergence if Re{a+b-c} > 1.

Since a finite linear combination of convergent series is convergent, we have immediately that the series in Eqs. (5) and (6) are convergent for Re{a+b-c}< 1 and 0< 0< π . Moreover, use of the triangle inequality shows that these series converge absolutely for Re{a+b-c}< 0.

By Eq. (4), one sees that

Re{a+b-c}=Re{
$$(\mu + \frac{1}{2}) + (\mu + \nu + 1) - (\nu + \frac{3}{2})$$
}
=Re{2\mu}
=2Re{\mu}.

Therefore, the series in Eqs. (5) and (6) converge absolutely for $\text{Re}\{\mu\}<0$ and at least conditionally for $\text{Re}\{\mu\}<\frac{1}{2}$ if $0<\theta<\pi$. It should be noted that we have assumed that a,b, and c are not negative integers, so that the expressions involving the gamma function are meaningful; this assumption of course imposes additional discrete restrictions on the values of μ and ν .

Continuing the analysis, we will develop an asymptotic expansion for large k of

$$w(a,b,c,k) = \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(1+k)\Gamma(c+k)}.$$
 (8)

To this end (see Appendix C), we observe that

$$\log \Gamma(k+x) = \log \Gamma((k) + x \log k + O(\frac{1}{k}), \qquad (9)$$

for large k. Using Eq. (9) in Eq. (8), we have

log w(a,b,c,k) = log
$$\Gamma(k+a) + \log \Gamma(k+b) - \log \Gamma(k+c) - \log \Gamma(k+1)$$

= $[\log \Gamma(k) + a \log k + O(\frac{1}{k})]$
+ $[\log \Gamma(k) + b \log k + O(\frac{1}{k})]$
- $[\log \Gamma(k) + c \log k + O(\frac{1}{k})]$
- $[\log \Gamma(k) + \log k + O(\frac{1}{k})]$
= $(a+b-c-1) \log k + O(\frac{1}{k})$.

Thus

$$w(a,b,c,k) = k^{a+b-c-1} [1 + O(\frac{1}{k})],$$
 (10)

and so

$$w(a,b,c,k) \sim k^{a+b-c-1} \text{ as } k \to \infty^{6}.$$
 (11)

For real values of x and y,

$$|\sin(x+iy)| = (\sin^2 x + \sinh^2 y)^{\frac{1}{2}},$$
 (12)

and

$$|\cos(x+iy)| = (\cos^2 x + \sinh^2 y)^{\frac{1}{2}}$$
 (13)

Since $\sin^2 x \le \text{ and } \cos^2 x \le 1 \text{ and } 1 + \sinh^2 y = \cosh^2 y$,

$$(\sin^2 x + \sinh^2 y)^{\frac{t}{2}} \leq \cosh y$$
,

and
$$(\cos^2 x + \sinh^2 y)^{\frac{1}{2}} \le \cosh y$$
.

Also, $\sin^2 x \ge 0$ and $\cos^2 x \ge 0$, so that

$$(\sin^2 x + \sinh^2 y)^{\frac{1}{2}} \ge |\sinh y|$$

and
$$(\cos^2 x + \sinh^2 y)^{\frac{1}{2}} \ge |\sinh y|$$

This shows that

$$|\sinh(\theta \cdot Im\{b\})| \le |\sin[(2k+b)\theta]| \le \cosh(\theta \cdot Im\{b\})$$
, (14)

⁶F. W. J. Olver, Asymptotics and Special Functions, 1974, pg. 301.

and

$$|\sinh(\theta \cdot \operatorname{Im}\{b\})| \leq |\cos[(2k+b)\theta]| \leq \cosh(\theta \cdot \operatorname{Im}\{b\}).$$
 (15)

For fixed θ and b, the upper bound, $\cosh(\theta \cdot \text{Im } \{b\})$, is finite and independent of k. If $0 < \theta < \pi$ and Im $\{b\} \neq 0$,

$$|\sinh(\theta \cdot Im \{b\})| = \rho > 0$$
. If $0 < \theta < \pi$ and $Im \{b\} = 0$,

 $\left|\sin[(2k+b)\theta]\right|$ and $\left|\cos[(2k+b)\theta]\right|$ can be neither constantly zero nor remain arbitrarily close to zero as k varies. Therefore, the expressions

$$|\mathbf{k}|^{\text{Re}\{\mathbf{a}+\mathbf{b}-\mathbf{c}-1\}}|\sin[(2\mathbf{k}+\mathbf{b})\theta]|, \tag{16}$$

and

$$|\mathbf{k}|^{\text{Re}\{\mathbf{a}+\mathbf{b}-\mathbf{c}-1\}}|\cos[(2\mathbf{k}+\mathbf{b})\theta]| \tag{17}$$

will tend to zero as $k\rightarrow\infty$ only if $Re\{a+b-c-1\}<0$.

Using Eqs. (8), (11), (16), and (17), we see that the kth term of the series in Eqs. (5) and (6) cannot tend to zero in absolute value as $k\rightarrow\infty$ when $Re\{a+b-c-1\}\geq 0$. Since

$$a+b-c-1 = 2\mu-1$$
,

we conclude that the series in Eqs. (5) and (6) diverge for $Re\{\mu\}_{\geq \frac{1}{2}}$.

III. NUMERICAL ASPECTS OF CONVERGENCE

To illustrate numerically the convergence and divergence properties obtained, use was made of the following formulae 2 , 4

$$P_{\nu}^{1}(\cos\theta) = -\frac{\Gamma(\nu+2)}{\Gamma(\nu)} \sqrt{\frac{2}{\pi}} \frac{(\sin\theta)^{-1}}{\Gamma(\frac{3}{2})} \int_{0}^{\theta} \cos[(\nu+\frac{1}{2})\phi](\cos\phi-\cos\theta)^{\frac{1}{2}} d\phi; (18)$$

$$P_{\nu}^{\frac{1}{2}}(\cos\theta) = \left(\frac{\pi}{2}\right)^{-\frac{1}{2}}(\sin\theta)^{-\frac{1}{2}}\cos(\nu + \frac{1}{2})\theta; \tag{19}$$

$$Q_{\nu}^{\frac{1}{2}}(\cos\theta) = -\frac{\pi}{2} e^{-\frac{1}{2}}(\sin\theta)^{-\frac{1}{2}}\sin(\nu + \frac{1}{2})\theta;$$
 (20)

$$P_{\nu}^{-\frac{1}{2}}(\cos\theta) = \frac{\pi}{2}^{-\frac{1}{2}}(\nu + \frac{1}{2})^{-1}(\sin\theta)^{-\frac{1}{2}}\sin(\nu + \frac{1}{2})\theta; \tag{21}$$

$$Q_{\nu}^{-\frac{1}{2}}(\cos\theta) = (2\pi)^{\frac{1}{2}}(2\nu+1)^{-1}(\sin\theta)^{-\frac{1}{2}}\cos(\nu+\frac{1}{2})\theta.$$
 (22)

These associated Legendre functions were computed for

 $\nu=-.4,.05,-1.5$, and 1.5, with $\theta=\frac{\pi}{6}$. From the recurrence formula⁵

$$T_{\nu}^{\mu+2}(\cos\theta) + 2(\mu+1)(\cos\theta)(1-\cos^{2}\theta)^{-\frac{1}{2}}T_{\nu}^{\mu+1}(\cos\theta) + (\nu-\mu)(\nu+\mu+1)T_{\nu}^{\mu}(\cos\theta) = 0,$$
(23)

where T stands for either P or Q, one obtains values for $P_{\nu}^{\frac{3}{2}}(\cos\theta)$ from Eqs. (19) and (21), and values for $Q_{\nu}^{\frac{3}{2}}$ (cos θ) from Eqs. (20) and (22).

With ν and θ as above, the sums of the first fifty terms of the series in Eqs. (1) and (2) were computed for $\mu=\pm\frac{1}{2},\pm1,0$, and $\frac{3}{2}$. The values of $P_{\nu}^{-\frac{1}{2}}$ and $Q_{\nu}^{-\frac{1}{2}}$ thus computed agree quite well with those obtained from Eqs. (21) and (22), while the values for $P_{\nu}^{\frac{1}{2}}$, $Q_{\nu}^{\frac{1}{2}}$, and P_{ν}^{1} do not agree with the values obtained from Eqs. (13) - (20), as expected, since these values of μ are outside the region of convergence of the series. Indeed, one sees that the terms of the sequence of partial sums increase quite rapidly for $\mu=\frac{1}{2}$ and 1. This growth behavior is again exhibited in the partial sums when $\mu=\frac{3}{2}$. Using the series approximations for the values $\mu=-1$ and 0 in the recurrence relation (23), the value of P_{ν}^{1} obtained agrees well with that computed from Eq. (18). The results are tabulated in Appendix A.

IV. CONCLUSIONS

At θ =0 or π , the term (sin θ) $^{\mu}$ is meaningless, so the series representations in Eqs. (1) and (2) are not valid there. Indeed, ± 1 are singular points of the associated Legendre functions. The series in Eqs. (5) and (6) do converge absolutely for $0 \le \theta < \pi$ when $Re\{\psi\} < 0$, however.

V. SUMMARY

It is shown that the series representations

$$P_{\nu}^{\mu}(\cos\theta) = \pi^{-\frac{1}{2}}2^{\mu+1}(\sin\theta)^{\mu} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \sum_{k=0}^{\infty} \frac{(\mu+\frac{1}{2})_{k}(\mu+\nu+1)_{k}}{k! (\nu+\frac{3}{2})_{k}} \sin[(2k+\mu+\nu+1)\theta]$$

and

$$Q_{\nu}^{\mu}(\cos\theta) = \pi^{-\frac{1}{2}} 2^{\mu} (\sin\theta)^{\mu} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \sum_{k=0}^{\infty} \frac{(\mu + \frac{1}{2})_{k} (\mu + \nu + 1)_{k}}{k! (\nu + \frac{3}{2})_{k}} \cos[(2k + \mu + \nu + 1)\theta],$$

for the associated Legendre functions of the first and second kind, respectively, of complex order μ and degree ν with $0<\theta<\pi$, converge absolutely for $Re\{\mu\}<0$, converge at least conditionally for $0\le Re\{\mu\}<\frac{1}{2}$, and diverge for $Re\{\mu\}>\frac{1}{2}$.

Numerical computations for real μ and ν are presented to illustrate the behavior of the partial sums in the regions of convergence and divergence. A method is indicated whereby these series representations may be used to compute the values of $P^{\mu}_{\nu}(\cos\theta)$ for arbitrary complex μ and $\nu.$

VI. ACKNOWLEDGMENTS

Mr. A. S. Elder verified a large portion of the analysis, and provided the authors with a great deal of insight to the problem.

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APPENDIX A

NUMERICAL ASPECTS OF CONVERGENCE AND DIVERGENCE

Tables of the first 50 partial sums (PSP) of $P_{\nu}^{\mu}(\cos\theta)$ and the partial sums (QSP) of $Q_{\nu}^{\mu}(\cos\theta)$ for ν =-1.5, -.4, .05, 1.5; μ =± 1.0, ± .5, 0, 1.5; and θ = $\pi/6$ are given in this appendix. The functional values PUV (COS THETA) = $P_{\nu}^{\mu}(\cos\theta)$ and QUV (COS THETA) = $Q_{\nu}^{\mu}(\cos\theta)$, are listed at the bottom of each table. A discussion of these equations begins on page 12 of the text.

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THETA = 30.0 DEGREES
     MU = -1.00
                    NU =-1.50
              PSP
                                         USP
                                 .167108551642066592E-99
    -.167108551642066592E-99
     .4587448131796507H0E+00
                                 .171205895049514944E+01
 5
                                 .165471584884769308E+01
 3
     .244737444367757088E+00
     .225154410972202421E+00
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27
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28
     .231399176218350930E+00
                                 .168145924979684784E+01
29
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                                 .168147596605840063E+01
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                                 .168145574703557864E+01
     .231405334165886294E+00
                                 .168146169697865135E+01
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                                 .168146911302856790E+01
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     .231414106352948634E+00
                                 .168147093108057898E+01
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37
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                                 .168146637503109881E+01
38
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41
     .231409497400540590E+00
                                 .168146731289217321E+01
42
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                                 .168146843293875515E+01
     .231416520255068183E+00
43
                                 .168146559015497821E+01
44
     .231415551867065736E+00
                                 .168146197615639265E+01
     .231412182864864217E+00
45
                                 .168146107342961447E+01
                                 .168146337623954573E+01
46
     .231409880054932955E+00
47
                                 .168146631792201372E+01
     .231410668276374641E+00
48
     .231413423212349226E+00
                                 .168146705610488331E+01
49
     .231415314588848292E+00
                                 .168146516472838424E+01
50
     .231414664461460293E+00
                                 .168146273841994084E+01
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PUV (COS THETA) = .261123486338710080E+00
QUV (COS THETA) = .298031510586655852E+01

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                    NU = -.40
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              PSP
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                                -.382784634055499464E+01
     .284871085053336490E+00
                                -. 430394647866439731E+U1
     .236535642182184539E+00
                                -.428824134124941962E+01
 3
     .233808758707531680E+00
                                -.427541236314952855E+01
     .237671811439941889E+00
 5
                                -.427193405484593058E+U1
     .240114644948309452E+00
                                -.427272777956741693E+01
 6
     .240416678864225650E+00
 7
                                --427414873742231996E+01
     .239747650409969615E+00
                                -.427475113334828242E+01
 8
                                -.427456695618430531E+01
 9
     .239180811384473643E+00
     .239094568498430055E+00
                                -.427416121530589633E+01
10
     .239317675825103320E+00
                                -. 427396032856664725E+01
11
     .239530943912509137E+00
                                -.427402962356882332E+01
12
     .239566681646089349E+00
13
                                -.427419775638626930E+01
     .239466621130497154E+00
                                -.427428785127918304E+01
14
     .239364449808307819E+00
15
                                -.427425465380421357E+01
     .239346344493988721E+00
                                -.427416947499732637E+01
16
     .239399527834941158E+00
                                -.427412158850204352E+01
17
                                -.427413998333238955E+01
     .239456141301466267E+00
18
                                -.427418893401686653E+01
     .239466546090650086E+00
19
                                -.427421735434045360E+01
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     .239434982123598783E+00
     .2394004054613n1019E+00
                                -.427420611970184307E+01
21
                                -.427417544572612532E+01
22
     .239393885506489706E+00
     .239414127126163610E+00
                                -.427415722008990780E+01
23
     .239436769946629256E+00
                                -.427416457718825536E+01
24
                                -.427418505064920611E+01
25
     .239441121715094434E+00
                               -.427419742903176908E+01
     .239427374128506652E+00
26
27
     .239411748169980640E+00
                                -.427419235185007146E+01
     .239408700511189855E+00
                                -.427417801374276092E+01
28
     .239418459932122391E+00
29
                                -.427416922632068326E+01
     .239429693241146503E+00
                                -.427417287624403906E+01
30
     .239431909923346422E+00
                                -.427418330491385995E+01
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                               -.427418976632034093E+01
32
     .239416389526758239E+00
                                -.427418705510016959E+01
33
     .239414727197549519E+00
                                -.427417923445612238E+01
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35
     .239420156940233186E+00
                                -.427417434549385051E+01
     .239426523737773247E+00
36
                               -.427417641419177320E+01
     .239427802174138609E+00
                               -.427418242876199073E+01
37
     .239423595136660236E+00
                               -.427418621679555077E+01
38
                               -.427418460262037355E+01
39
     .239418627216291188E+00
40
     .239417623005326797E+00
                               -.427417987H17923421E+01
41
     .239420948552991952E+00
                               -.427417688384266700E+01
42
     .239424899149243493E+00
                                -.427417816746920099E+01
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     .239425702277493777E+00
                               -.427418194589054905E+01
     .239423028192837342E+00
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44
     .239419835072523379E+00
                               -.427418331613950605E+01
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     .239419182731699194E+00
                               -.427418024711722295E+01
46
     .239421364985298508E+00
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     .239423982538757361E+00
48
                               -.427417913270193117E+01
     .239424519596152021E+00
49
                               -.427418165935832060E+01
                                -.427418328369359554E+01
     .239422715589067396E+00
50
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      PUV (COS THETA) =
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-.757579262065869710E+01

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THETA = 30.0 DEGREES
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                    NU =
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                                         QSP
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     .573389094124100302E+00
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     .262906521267552242E+00
                                 .217282677997457212E+02
 5
     .231897101684802442E+00
                                 .217472703965057921E+02
 3
 4
                                 .217577676217403527E+02
     .232171981201488736E+00
 5
                                 .217598668261245825E+02
     .236038234720196846E+00
     .237954332598154616E+00
                                 .217586926398348167E+02
 6
 7
                                 .217573936241425466E+02
     .237920316675119689E+00
 8
     .237201306854259488E+00
                                 .217570032336622145E+02
     .236733950151301496E+00
 9
                                 .217572896302181297E+02
     .236744029047381258E+00
                                 .217576745277308306E+02
10
                                 .217578084867314195E+02
     .236990750834588045E+00
11
                                 .217576982543736424E+02
     .237170633693602696E+00
12
                                 .217575357736843683E+02
13
     .237166378987016857E+00
                                 .217574747616169581E+02
14
     .237054008737564049E+00
                                 .217575282632991627E+02
15
     .236966701927194290E+00
     .236968881592398611E+00
                                 .217576115013550991E+02
16
     .237029186936843972E+00
                                 .217576442444855843E+02
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18
     .237077973899013188E+00
                                 .217576143477967161E+02
19
     .237076711929352986E+00
                                 .217575661551197134E+02
     .237040683668720124E+00
                                 .217575465933702547E+02
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                                 .217575649610703241E+02
     .237010710306373188E+00
21
     .237011505313664511E+00
                                 .217575953211739097E+02
55
     .237034721310486339E+00
                                 .217576079264317069E+02
23
                                 ·217575958454496696E+02
     .237054435681406054E+00
24
     .237053902927230996E+00
                                 .217575755003885401E+02
25
     .237038077392936756E+00
                                 .217575669078244952E+02
56
                                 .217575752731129158E+02
     .237024426482914202E+00
27
28
     .237024800746327102E+00
                                 .217575895656560257E+02
29
     .237036067683244332E+00
                                 .217575956831036424E+02
     .237045907414126316E+00
                                 .217575896533088028E+02
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31
     .237045634519504451E+00
                                 .217575792318836434E+02
32
     .237037330661744143E+00
                                 .217575747232567434E+02
33
     .237030005661296723E+00
                                 .217575792120227907E+02
34
     .237030210726988579E+00
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                                 .217575904611080975E+02
35
     .237036505793938648E+00
                                 .217575870299460939E+02
     .237042104941443016E+00
36
     .237041946964984809E+00
                                 .217575809970685498E+02
37
     .237037061635952892E+00
                                 .217575783445513073E+02
38
     .237032686070360965E+00
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39
40
     .237032810339205796E+00
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42
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                                 .217575857361094229E+02
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     .237036948762247186E+00
                                 .217575802457790149E+02
44
45
     .237034129579724903E+00
                                 .217575819733762865E+02
     .237034210491307840E+00
                                 .217575850632649781E+02
46
     .237036753614958012E+00
47
                                 .217575864440684588E+02
     .237039066890734780E+00
                                 .217575850264912397E+02
48
     .237039000215777207E+00
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49
     .237036896003740911E+00
                                 .217575813377834179E+02
50
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PUV (COS THETA) = .267467495283606802E+00 QUV (COS THETA) = .385643088285442081E+02

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MU =-1.00
                    NU = 1.50
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     .313328534328874664E+00
 1
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                                 .342000085152602839E+00
 5
     .201844920106720291E+00
 3
                                 .358719410841032034E+00
 4
     .206128708661997875E+00
                                 .363003199396309617E+00
 5
                                 .362268210910838070E+00
     .208871723032755789E+00
     .209275966699765140E+00
                                 .360759553006921217E+00
 6
 7
                                 .360086550581305708E+00
     .208602964274149631E+00
 A
     .208001016505566735E+00
                                 .360247841999783211E+00
 9
     .207889624619680709E+00
                                 .360663562177460774E+00
10
     .208108648769076134E+00
                                 .360882586326856198E+00
     .208331173229712727E+00
                                 .360822961077332452E+00
11
12
     .208376726086429295E+00
                                 .360652955501636313E+00
13
     .208279497399726506E+00
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     .208173754748978007E+00
                                 .360584060472807111E+00
14
     .208150828519113776E+00
15
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     .208202226575947327E+00
                                 .360721020384319973E+00
16
     .208260552944117619E+00
                                 .360705391881071303E+00
17
     .208273677499864374E+00
18
                                 .360656410372197644E+00
     .208243273791692187E+00
                                 .360626006664025457E+00
19
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20
     .208199549085846767E+00
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51
25
     .208219003814293076E+00
                                 .360685595887711775E+00
     .208242219761406788E+00
23
                                 .360679375193431133E+00
     .208247685858424691E+00
24
                                 .360658975441641216E+00
     .208234493266769126E+00
25
                                 .360645782849985651E+U0
     .208218494254703280E+00
                                 .360650069772348390E+00
26
27
     .208214671061923048E+00
                                 .360664338122051344E+00
     .208224025680431426E+00
                                 .36067369274U559722E+00
28
                                 .360670614172512493E+00
29
     .208235515052798240E+00
     .208238293126578344E+00
                                 .36066024626U017969E+0U
30
     .208231420764804896E+00
                                 .360653373898244521E+00
31
     .208222893264856416E+00
                                 .360655658834969172E+00
32
33
     .208220811515289954E+00
                                 .360663428030119844E+00
     .208226007772391891E+00
                                 .360668624287221781E+00
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     .208232510123931555E+00
                                 .360666881987377825E+00
35
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                                 .360660910619371225E+00
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37
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     .208223759122058191E+00
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39
     .208226938367947760E+00
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41
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                                 .360661285062481995E+00
42
     .208229417945309924E+00
                                 .360658729623850583E+00
43
                                 .360659602259736157E+0U
44
     .208226161223848456E+00
45
     .208225345822357057E+00
                                 .360662645379530624E+00
46
     .208227430555903794E+00
                                 .360664730113077361E+00
     .208230099400129173E+00
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                                 .360664014998422446E+00
     .208230770499368398E+00
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                                 .360661510421964739E+00
                                 .360659787526340995E+00
     .208229047603744655E+00
49
                                 .360660380865140676E+00
50
     .208226833233198145E+00
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PUV (COS THETA) = .234958820650612321E+00
QUV (COS THETA) = .639253880933468677E+00

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                    NU =-1.50
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    -.4999999999999360E+00
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 5
    0.
     .4999999999999701-100
                                 .173205080756887508E+01
 3
                                 .173205080756887508E+01
 4
     .583333333333332984-100
 5
                                 .173205080756887508E+01
     .54166666666666342-100
 6
     .49166666666666372-100
                                 .173205080756887508E+01
 7
     .47499999999999716-100
                                 .173205080756887508E+01
 8
     .486904761904761613-100
                                 .173205080756887508E+01
     .504761904761904460-100
 9
                                 .173205080756887508E+01
10
                                 .173205080756887508E+01
     .511706349206348900-100
                                 .173205080756887508E+01
11
     .506150793650793348-100
     .497059884559884262-100
                                 .173205080756887508E+U1
15
     .493272005772005476-100
                                 .173205080756887508E+01
13
     .496477133977133679-100
                                 .173205080756887508E+U1
14
     .501971639471639171-100
                                 .173205090756887508E+01
15
     .504352591852591550-100
                                 •173205080756887508E+01
16
     .502269258519258218-100
                                 .173205080756887508E+01
17
     .498592787931022926-100
                                 .173205080756887508E+01
18
                                 .173205080756887508E+01
19
     .496958801002918352-100
                                 .173205080756887508E+U1
50
     .498420789307011918-100
                                 .173205080756887508E+01
51
     .501052368254380338-100
                                 .173205080756887508E+01
22
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                                 .173205080756887508E+01
23
     .501160593362605446-100
     .499184308777625210-100
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24
                                 .173205080756887508E+01
25
     .498278511676175935-100
     .499111845009509268-100
                                 .173205080756887508E+01
26
     .500650306547970806-100
                                 .173205080756887508E+01
27
28
     .501362557260221517-100
                                 .173205080756887508E+01
     .500701181598845856-100
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                                 .173205080756887508E+01
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     .499469654505249798-100
                                 .173205080756887508E+01
     .498894941861571638-100
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                                 .173205080756887508E+01
32
     .499432576270173788-100
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                                 .173205080756887508E+01
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                                 .173205080756887508E+01
34
     .500468492836213693-100
                                 .173205080756887508E+U1
35
36
     .499628156701759912-100
                                 .173205080756887508E+01
37
     .499231331304934515-100
                                 .173205080756887508E+01
                                 .173205080756887508E+U1
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                                 .173205080756887508E+01
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43
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     .499711552113468307-100
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44
45
     .500240093339683951-100
                                 .173205080756887508E+01
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                                 .173205080756887508E+U1
46
47
     .500251072698489397-100
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48
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                                .173205080756887508E+01
49
     .499566906802713881-100
                                 .173205080756887508E+01
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      PUV (COS THETA) =
                           .563940766730145087-100
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.306998012383946154E+01

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THETA = 30.0 DEGREES
     MU = -.50
                    NU = -.40
                                         QSP
              PSP
 K
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     .523359562429437657E+00
 2
                                 .998629534754572595E+01
                                 .998629534754572595E+01
     .523359562429437657E+00
 3
 4
                                 .998629534754572595E+01
     .523359562429437657E+00
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     .523359562429437657E+00
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     .523359562429437657E+00
                                 .998629534754572595E+01
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10
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                                 .998629534754572595E+01
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28
     .523359562429437657E+00
                                 .998629534754572595E+01
29
     .523359562429437657E+0n
                                 .998629534754572595E+Ul
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                                 .998629534754572595E+01
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                                 .998629534754572595E+01
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     .523359562429437657E+00
                                 .998629534754572595E+Ul
     .523359562429437657E+00
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                                 .998629534754572595£+01
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     .523359562429437657E+00
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                                 .998629534754572595E+01
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     .523359562429437657E+00
                                 .998629534754572595E+01
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44
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                                 .998629534754572595E+Ul
45
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                                 .998629534754572595E+01
46
     .523359562429437657E+00
                                 .998629534754572595E+01
47
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                                 .998629534754572595E+01
48
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49
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50
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PUV (COS THETA) = .590548027145600779E+00 QUV (COS THETA) = .177002476450372605E+02

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     .516391535825313189E+00
 1
                                 .174330860885125786E+01
 2
     .516391535825313189E+00
                                 .174330860885125786E+01
     .516391535825313189E+00
 3
     .516391535825313189E+00
                                 .174330860885125786E+01
                                 .174330H608B5125786E+01
 5
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     .516391535825313189E+00
                                 .174330860885125786E+01
 6
 7
      516391535825313189E+00
                                 .174330860885125786E+U1
                                 .174330860885125786E+01
 8
     .516391535825313189E+00
                                 .174330860885125786E+01
 9
      .516391535825313189E+00
      .516391535825313189E+00
                                 .174330860885125786E+01
10
                                 .174330860885125786E+01
     .516391535825313189E+00
11
                                 .174330860885125786E+01
     .516391535825313189E+00
12
                                 .174330860885125786E+01
     .516391535825313189E+00
13
     .516391535825313189E+00
                                 .174330860885125786E+01
14
                                 .174330860885125786E+01
15
      516391535825313189E+00
     .516391535825313189E+00
                                 .174330860885125786E+01
16
                                 .174330860885125786E+01
17
     .516391535825313189E+00
                                 .174330860885125786E+01
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19
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                                 .174330860865125786E+01
                                 .174330860885125786E+01
     .516391535825313189E+00
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      516391535825313189E+00
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22
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                                 .174330860885125786E+01
23
      .516391535825313189E+00
24
     .516391535825313189E+00
                                 .174330860885125786E+01
25
      516391535825313189E+00
                                 .174330860885125786E+01
26
     .516391535825313189E+00
                                 .174330860885125786E+01
                                 .174330860885125786E+01
27
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                                 .174330860885125786E+01
     .516391535825313189E+00
28
29
                                 .174330860885125786E+01
     .516391535825313189E+00
     .516391535825313189E+00
                                 .174330860885125786E+01
30
     .516391535825313189E+00
                                 .174330860885125786E+01
31
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                                 .174330860885125786E+01
32
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     .516391535825313189E+00
                                 .174330860885125786E+01
     .516391535825313189E+00
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35
                                 .174330860885125786E+01
     .516391535825313189E+00
                                 .174330860885125786E+01
36
     .516391535825313189E+00
                                 .174330860885125786E+01
37
     .516391535825313189E+00
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38
39
     .516391535825313189E+00
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     .516391535825313189E+00
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     .516391535825313189E+00
                                 .174330860885125786E+01
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42
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                                 .174330860885125786E+01
     .516391535825313189E+00
                                 .174330860885125786E+01
43
     .516391535825313189E+00
                                 .174330860885125786E+01
44
45
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     .516391535825313189E+00
                                 .174330860885125786E+01
46
47
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                                 .174330860885125786E+01
     .516391535825313189E+00
                                 .174330860885125786E+01
48
     .516391535825313189E+00
49
                                 .174330860885125786E+01
     .516391535825313189E+00
                                 .174330860885125786E+01
50
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PUV (COS THETA) = QUV (COS THETA) =

.582685451089739438E+00

.308993405707514995E+01

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                    NU = 1.50
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     .433012701892218769E+00
 1
                                .2499999999999680E+00
 5
     .433012701892218769E+00
 3
     .433012701892218769E+00
                                .24999999999999680E+00
     .433012701892218769E+00
                                .24999999999999680E+00
 4
     .433012701892218769E+00
                                .24999999999999680E+00
 5
                                .24999999999999680E+00
     .433012701892218769E+00
 6
     .433012701892218769E+00
                                .2499999999999680E+00
 7
                                .24999999999999680E+00
 8
     .433012701892218769E+00
                                .2499999999999680E+0U
 9
     .433012701892218769E+00
                                .24999999999999680E+00
10
     .433012701892218769E+00
                                .2499999999999680E+00
     .433012701892218769E+00
11
15
     .433012701892218769E+00
                                .2499999999999680E+00
     .433012701892218769E+00
13
                                .2499999999999680E+00
     .433012701892218769E+00
                                .24999999999999680E+00
14
                                .2499999999999680E+0U
     .433012701892218769E+00
15
                                .24999999999999680E+0U
     .433012701892218769E+00
16
                                .2499999999999680E+00
     .433012701892218769E+00
17
                                .24999999999999680E+00
     .433012701892218769E+00
18
19
     .433012701892218769E+00
                                .24999999999999680E+0U
     .433012701892218769E+00
20
                                .24999999999999680E+00
15
     .433012701892218769E+00
                                .2499999999999680E+00
55
     .433012701892218769E+00
                                .24999999999999680E+00
     .433012701892218769E+00
                                .2499999999999680E+00
23
                                .2499999999999680E+00
24
     .433012701892218769E+00
                                .2499999999999680E+00
25
     .433012701892218769E+00
                                .2499999999999680E+00
     .433012701892218769E+00
26
27
                                .24999999999999680E+00
     .433012701892218769E+00
28
     .433012701892218769E+00
                                .24999999999999680E+00
29
     .433012701892218769E+00
                                .24999999999999680E+00
                                .24999999999999680E+00
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     .433012701892218769E+00
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     .433012701892218769E+00
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     .433012701892218769E+00
                                .24999999999999680E+00
33
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     .433012701892218769E+00
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     .433012701892218769E+00
37
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     .433012701892218769E+00
                                .24999999999999680E+00
38
     .433012701892218769E+00
                                .24999999999999690E+00
39
     .433012701892218769E+00
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                                .2499999999999680E+00
41
     .433012701892218769E+00
     .433012701892218769E+00
42
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     .433012701892218769E+00
43
                                .24999999999999680E+00
     .433012701892218769E+00
44
                                .2499999999999680E+00
45
     .433012701892218769E+00
                                .24999999999999680E+00
     .433012701892218769E+00
                                .24999999999999680E+00
46
     .433012701892218769E+00
                                .2499999999999680E+00
47
     .433012701892218769E+00
48
                                .24999999999999680E+00
49
     .433012701892218769E+00
                                .2499999999999680E+00
50
                                .2499999999999680E+00
     .433012701892218769E+00
      PUV (COS THETA) =
                           .488602511902919296E+00
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.443113462726378440E+00

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     .626657068657749328E+00
 2
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                                 .540642416186564807E+00
 3
     .100142727967008017E+01
                                 .340010507925414482E+00
     .894332565788140590E+00
                                 .232915794043474900E+00
 5
     .779125962216308178E+00
                                 .263785310433279892E+00
 6
 7
     .753658611194719059E+00
                                 .358830758380041632E+00
     .812882824648883851E+00
 8
                                 .418054971834206424E+00
     .883310713573082701E+00
 9
                                 .399183875872338531E+00
                                 .336825849220704133E+00
10
     .900019496455986565E+00
                                 .295868333283759793E+00
     .859061980519042225E+00
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     .808326403493899003E+00
15
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     .795890473610275832E+00
13
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14
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                                 .376556927755922800E+00
15
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     .876755736060608748E+00
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     .818870380602645191E+00
                                 .322975669328277894E+00
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19
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                                 .322959408799884872E+00
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25
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27
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                                 .342403431402542377E+00
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     .845390140500710601E+00
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     .829706468587040199E+00
                                 .335539889116441878E+00
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37
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                                 .350784791637289358E+00
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38
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     .827774861544599941E+00
43
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44
     .837107323426516310E+00
                                 .356303307612054517E+00
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                                 .344124742194596117E+00
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                                 .335404301768595419E+00
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49
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THETA = 30.0 DEGREES
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                                 .139620160437844438E+01
     .106453580259884789E+01
 3
     .101184759965827758E+01
                                 .123404398958490236E+01
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     .883714269611804624E+00
                                 .128583283411624536E+01
     .812560358102438076E+00
 6
                                 .137093955168973297E+01
 7
     .840213206923795422E+00
                                 .138699924801884325E+01
     .915768137822808203E+00
 8
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      961232297012808759E+00
 9
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      .942492101917731445E+00
10
                                 .126744450335937297E+01
      .888928521147301482E+00
11
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     .855527818445381587E+00
12
      .869699333613550190E+00
                                 .134815518071411635E+01
13
     .911185587999326050E+00
                                 .135697335630292474E+01
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      926187890244118656E+00
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                                 .130962939741732848E+01
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                                 .133863325845913115E+01
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                                 .132269195088200570E+01
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                                 .130303793673931732E+01
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                                 .129889315874401221E+01
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                                 .132167785423058111E+01
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33
     .912503517325865415E+00
                                 .130556489130241440E+01
34
                                 .130214510647595650E+01
     .896414694663373998E+00
35
     .885720433174596434E+00
                                 .131402228712224873E+01
36
     .890523193520703898E+00
37
                                 .132880366357246180E+01
     .905318067229576274E+00
                                 .133194841105836192E+01
38
39
     .915174693816573956E+00
                                 .132100151821684414E+01
     .910738541569839300E+00
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                                 .130443778930739214E+01
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42
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     .892025848343324674E+00
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     .904770694948813094E+00
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                                 .130867427337701745E+01
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                                 .130614079672900841E+01
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48
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                                 .131500483368340006E+01
49
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                                 .132611392851184789E+01
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                                 .132849324215246377E+01
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PUV (COS THETA) = .102044614584513939E+01 QUV (COS THETA) = .235469296295508863E+01

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MU = 0.00
                                   THETA =
                                            30.0 DEGREES
                    NU = .05
             PSP
                                         OSP
                                 .93382652114909391AE+00
     .572249628146135628E+00
 1
                                 .924115899393766413E+00
     .943083311798271430E+00
 5
     .104980823921997599E+01
                                 .727553035254437101E+00
 3
     .966136593978889083E+00
                                 .591013319802927867E+U0
 5
     .841456810603975845E+00
                                 .594278174831646561E+00
 6
     .792720944061401262E+00
                                 .684038474935372513E+0U
 7
     .837906164610877192E+00
                                 .757774053322321263E+00
     .912864500982501054E+00
 A
                                 .755811200205137189E+00
                                 .697645316086848724E+00
     .944445999289034612E+00
 9
                                 .647137657148087749E+00
     .913494866084259005E+00
10
                                 .648540947939601095E+00
     .859905352810660753E+00
11
                                 .691564282997605580E+00
12
     .936545587823463039E+00
                                 .729973129949564509E+00
13
     .860082559507169495E+00
     .901784331829888204E+00
                                 .728881130610225706E+00
14
     .920319021779897419E+00
15
                                 .694744478278641805E+00
                                 .663758082738804572E+00
16
     .901330534171469489E+00
     .867199835350840669E+00
                                 .664651826541223099E+00
17
                                 .692944554144815117E+00
     .851838137640152650E+00
18
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     .867751437663766077E+00
19
                                 .718156283978244476E+00
20
     .896637786226762521E+00
21
                                 .693999052191680853E+00
      .909754092912764059E+00
     .896058742299643486E+00
                                 .671650271187839741E+00
55
     .871019739278238919E+00
                                 .672305940557128876E+00
53
     .859576117861378387E+00
                                 .693382469313491555E+00
24
25
     .871596157141802618E+00
                                 .712997390692606159E+00
     .893692217072239281E+00
                                 .712418785000279690E+00
26
     .903841529525588778E+00
27
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     .893131600345619522E+00
28
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     .873359442659461530E+00
29
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                                 .693560147202316581E+00
30
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                                 .709319494847421014E+00
31
     .891789308994330719E+00
                                 .708951013988572107E+00
32
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     .900066316590923118E+00
33
     .891273178076470877E+00
                                 .679257564453724432E+00
34
     .87493720835346089UE+00
                                 .679685336875647783E+00
35
36
     .867359181555885449E+00
                                 ·693642326005003592E+00
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37
     .875430083612281714E+00
     .890460033729010288E+00
                                 .706419268047439609E+00
38
     .897447942683905395E+00
39
                                 .693549140780061311E+00
                                 .681378293977736189E+00
     .889989638171110189E+00
40
                                 .681742731252897301E+00
     .876072340122301150E+00
41
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     .869589281080036237E+00
42
                                 .704995267453364286E+00
     .876521422581137450E+00
43
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     .889479451380565674E+00
44
45
     .895525692668211642E+00
                                 .693520158354011402E+00
46
     .889050367440208341E+00
                                 .682953387955986601E+00
     .876927896517358808E+00
47
                                 .683270826028797088E+00
48
     .871263324515495587E+00
                                 .693703669823669032E+00
49
     .877338318212939158E+00
                                 .703617158538139037E+00
                                 .703318949531016679E+00
50
     .888726460743439680E+00
                           .100282042354942522E+01
      PUV (COS THETA) =
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.124660n38051107279E+01

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                    NU = 1.50
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             PSP
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     .642022106435681038E+00
                                -.238010290131776227E-01
 5
     .837852440391227703E+00
     .790813177321048668E+00
                               -.199353948741684157E+00
 3
                                -.234633396044318433E+00
     -659148487524668768E+00
 5
                                -.157324399460793297E+00
     .581839490941143632E+00
 6
     .605487745461190670E+00
                                -.690679120816573948E-01
 7
                                -. 487451933534919712E-U1
     .681333164302635586E+00
                                -.974377200401378740E-01
 8
     .730025690989281489E+00
                                -.156677845359190553E+00
 9
     .714152347250522818E+00
                                -.170987905244889657E+00
10
     .660746476697740480E+00
                                -.135394427616732629E+00
11
     .625152999069583452E+00
                                -.907679639964962945E-01
     .637110623957682655E+00
12
     .678350257749597482E+00
                                -.797178374257974779E-01
13
                                -.107778597342953385E+00
14
     .706411017666753389E+00
     .696816390468573001E+00
                                -.143586233526244815E+00
15
     .663220402402484865E+00
                                -.152588251397478767E+00
16
     .640056810682505013E+00
                                -.129424659677498915E+UU
17
                                -.995214485101304327E-01
18
     .648069351965898728E+00
                                -. 919262270852468070E-01
     .676415104218300102E+00
19
                               -.111649632639973890E+00
     .696138509773027185E+00
20
21
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     .689259696382342815E+00
     .664743656230263911E+00
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22
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                                -.126717063882335857E+00
23
24
                                -.104226337533951885E+00
     .653596325204687006E+00
25
     .675194991493579786E+00
                                -.984389923442537508E-01
     .690403139973064413E+00
                                -.113647140823738378E+00
26
27
     .685041112098789741E+00
                                -.133658501282132000E+00
                               -.138830495601475482E+00
28
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                               -.125183965592947655E+00
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     .656922117780726066E+00
                                -.107159349721216200E+UU
30
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                                -.102484369182578917E+00
31
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                               -.114860250773818592E+00
35
33
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                               -.131257137139311174E+00
     .666433975237438855E+00
                               -.135522287157316267E+00
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                               -.124200504199074749E+00
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                               -.109161613874454937E+00
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                               -.105240208244719612E+00
37
38
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                               -.115673414725298233E+00
     .680488478460627951E+00
39
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                               -.133191057869579848E+00
40
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     .657271056035583868E+00
                               -.123517064676647708E+00
41
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                               -.110615089726213901E+00
42
                               -.107237988849044299E+0u
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     .682349445551023539E+00
44
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45
     .679121678747321806E+00
                               -.131459928562369280E+00
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     .667336062648046971E+00
47
     .658891066579714491E+00
                               -.123014932494036800E+00
48
     .661918059698237916E+00
                               -.111718040381546017E+00
                               -.108752533060742599E+00
49
     .672985483689693731E+00
     .680926107701536620E+00
                               -.116693157072585489E+00
50
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                    NU =-1.50
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             PSP
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                                .14999999999999808E+01
 2
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     .173205080756887508E+01
                                .99999999999998720E+00
 3
                                .252435489670723778E-28
     .173205080756887508E+01
                               -.4999999999999360E+00
 5
     .866025403784437538E+00
                               -.353409685539013289E-27
     .252435489670723778E-28
                                .9999999999998720E+00
 7
    -.277679038637795768E-27
                                .14999999999999808E+01
 8
     .866025403784437538E+00
 9
     .173205080756887508E+01
                                .99999999999998720E+00
10
     .173205080756887508E+01
                               -.100974195868289511E-27
                               -.49999999999999360E+U0
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11
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                               -.833037115913388467E-27
12
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    -.492249204857910591E-27
13
                                .1499999999999808E+01
     .866025403784437538E+00
14
                                .9999999999998720E+00
     .173205080756887508E+01
15
                                .227191940703651400E-27
     .173205080756887508E+01
16
                               -.4999999999999360E+U0
     .866025403784437538E+00
17
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                               -.492249204857911367E-27
18
                                .9999999999998720E+00
19
    -. 934011311781676943E-27
                                .14999999999999808E+01
20
     .866025403784437538E+00
                                .9999999999998720E+00
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                                .833037115913388467E-27
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     .866025403784437538E+00
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     .866025403784437538E+00
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                                .99999999999998720E+00
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28
     .173205080756887508E+01
                                .118644680145240176E-26
     .866025403784437538E+00
                               -.4999999999999360E+00
29
    -.126217744835361889E-27
                               -.593223400726200878E-27
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    -.133790809525483447E-26
31
                                .14999999999999808E+01
     .866025403784437538E+00
32
                                .9999999999998720E+UU
     .173205080756887508E+01
33
                                .156510003595848742E-26
34
     .173205080756887508E+01
35
     .866025403784437538E+00
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36
    -.169131778079384931E-26
37
    -.330690491468647942E-26
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38
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39
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41
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42
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                               -.130004277180422746E-26
43
    -.413994203059986685E-26
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44
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                                .99999999999998720E+00
45
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     .173205080756887508E+01
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46
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47
     .866025403784437538E+00
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49
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                                .99999999999998720E+00
                                .14999999999999808E+01
50
     .866025403784437538E+00
```

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MU =
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                    NU = -.40
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     .154326856976959898E+01
                                 .786334611702479190E+00
 3
     .199725906950914519E+01
                                -.104671912485887531E+00
 4
     .145262003449411881E+01
                                -.943342480431310487E+00
 5
     .453990499739546210E+00
                                -.891006524188366722E+00
 6
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                                -.391275008989621855E-27
     .544639035015026385E+00
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     .154326856976959898E+01
 8
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 9
     .199725906950914519E+01
10
     .145262003449411881E+01
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12
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                                 .838670567945422956E+00
13
     .154326856976959898E+01
                                 .786334611702479190E+U0
14
     .199725906950914519E+01
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                                -.891006524188366722E+00
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     .555358077275592311E-27
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28
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37
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                                 .786334611702479190E+00
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                               -.260008554360845491E-26
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                                 .838670567945422956E+00
50
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                                 .786334611702479190E+00
      PUV (COS THETA) =
                           .174139210336130303E+01
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.139374181061235289E+01

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.50
                                   THETA = 30.0 DEGREES
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                    NU =
                          .05
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                                 .688354575693753103E+00
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 2
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                                 .404339230989830849E+00
 3
     .191763946973638364E+01
                                -.568030689407844508E+00
                                -.125638526510159761E+01
 4
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                                -.972369920397675357E+00
 5
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 6
                                 .688354575693753103E+00
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     .725374371012286709E+00
 A
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 9
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                                -.125638526510159761E+01
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                                -.125638526510159761E+01
16
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17
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                                -.143888229112312553E-26
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                                -.125638526510159761E+01
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     .119226509872409694E+01
29
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                                -.381177589402792904E-26
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     .725374371012286709E+00
                                 .688354575693753103E+00
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     .168419410588047853E+01
                                 .404339230989830849E+00
35
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     .191763946973638364E+01
33
     .119226509872409694E+01
                                -.125638526510159761E+01
34
                                -.972369920397675357E+00
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    -.381808678126969714E-26
                                -.482151785271082415E-26
36
                                 .688354575693753103E+00
     .725374371012286709E+00
37
                                 .404339230989830849E+00
38
     .168419410588047853E+01
     .191763946973638364E+01
                                -.5680306894U7844508E+00
39
     .119226509872409694E+01
                                -.125638526510159761E+01
40
     .233445363855905113E+00
                                -.972369920397675357E+00
41
    -. 425353800095169565E-26
                                -.633613079073516682E-26
42
     .725374371012286709E+00
                                 .688354575693753103E+00
43
44
     .168419410588047853E+01
                                 .404339230989830849E+00
45
     .191763946973638364E+01
                                -.568030689407844508E+00
46
     .119226509872409694E+01
                                -.125638526510159761E+01
     .233445363855905113E+00
                                -.972369920397675357E+00
47
    -.320593071881819198E-26
                                -.744684694528635144E-26
48
                                 .688354575693753103E+00
49
     .725374371012286709E+00
50
     .168419410588047853E+01
                                 .404339230989830849E+00
      PUV (COS THETA) =
                           .190040954242058588E+01
```

.716672627040100654E+00

```
MU =
           .50
                   MU = 1.50
                                  THETA = 30.0 DEGREES
             PSP
     .9999999999998720E+00
                                .757306469012170364E-28
     .14999999999999808E+01
                               -.866025403784437538E+00
 2
     .9999999999998720E+00
                               -.173205080756887508E+01
 3
     .201948391736579022E-27
                               -.173205080756887508E+01
    -.4999999999999360E+00
                               -.866025403784437538F+00
 5
    -.315544362088404722E-27
                                .252435489670723778E-28
 6
     .9999999999998720E+00
                                .328166136571940523E-27
 7
     .1499999999999808E+01
                               -.866025403784437538E+00
      9999999999998720E+00
                               -.173205080756887508E+01
 9
                               -.173205080756887508E+01
     .403896783473158044E-27
10
    -.4999999999999360E+00
                               -.866025403784437538E+00
11
                               -.25243548967U723778E-27
    -.643710498660345633E-27
12
                                .151461293802433750E-27
     .99999999999998720E+00
13
     .14999999999999808E+01
                               -.866025403784437538E+00
14
      9999999999998720E+00
                               -.173205080756887508E+01
15
     .201948391736579022E-27
                               -.173205080756887508E+01
16
    -.4999999999999360E+00
17
                               -.866025403784437538E+00
18
    -.122431212490301032E-26
                                .328166136571940911E-27
     .9999999999998720E+00
                                .113595970351825597E-26
19
     .14999999999999808E+01
                               -.866025403784437538E+00
20
                               -.173205080756887508E+01
     .9999999999998720E+00
21
     .201948391736579022E-27
                               -.173205080756887508E+01
22
    -,4999999999999360E+00
                               -.866025403784437538E+00
23
    -.193113149598103690E-26
                                .68157582211U954200E-27
24
     .9999999999998720E+00
                                .148936938905726925E-26
25
     .14999999999999808E+01
                               -.866025403784437538E+00
26
27
     .99999999999998720E+00
                               -.173205080756887508E+01
     .126217744835361889E-27
                               -.173205080756887508E+01
28
    -.4999999999999360E+00
                               -.866025403784437538E+00
29
                                .111071615455118462E-26
    -.200686214288225403E-26
30
     .99999999999998720E+00
                                -272630328844381473E-26
31
     -14999999999999808E+01
                               -.866025403784437538E+00
32
      .99999999999998720E+00
                               -.173205080756887508E+01
33
    -.555358077275592311E-27
                               -.173205080756887508E+U1
34
    -.4999999999999360E+00
                               -.866025403764437538E+00
35
                                .272630328844381680E-26
    -.338263556158769862E-26
36
     .9999999999998720E+00
                                .434189042233644691E-26
37
     .14999999999999808E+01
                               -.866025403784437538E+00
38
     .9999999999998720E+00
                               -.173205080756887508E+01
39
    -.555358077275592311E-27
                               -.173205080756887508E+01
40
    -.4999999999999360E+00
                               -.866025403784437538E+00
41
                                .277679038637796156E-26
    -.478365252926021559E-26
42
     .9999999999998720E+00
                                .358458395332427661E-26
43
     .14999999999999808E+01
                               -.866025403784437538E+00
44
                               -.173205080756887508E+01
      9999999999998720E+00
45
    -.191850972149750071E-26
                               -.173205080756887508E+01
46
    -.49949999999999360E+00
                               -.866025403784437538E+00
47
48
    -.613418239899858780E-26
                                .206997101529993498E-26
     .9999999999998720E+00
49
                                .368555814919256509E-26
     .14999999999999808E+01
50
                               -.866025403764437538E+00
      PUV (COS THETA) =
                           .169256875064326669E+01
```

-.153499006191973077E+01

```
30.0 DEGREES
     MU = 1.00
                    NU =-1.50
                                   THETA =
              PSP
                                         QSP
     .458744813179651092-100
                                 .171205895049515060E-99
 1
                                 .344058609884738085E+00
 2
     .128404421287136208E+01
 3
     .304651721847128206E+01
                                -.141841439571518190E+01
     .210573195706770136E+01
                                -.492947279028531258E+01
 5
    -.250253218580559515E+01
                                -.616425344587751225E+01
 6
    -.667721800131595250E+01
                                -.198956763036715490E+01
 7
    -.485630240327233055E+01
                                 .480618189782630955E+01
     .303162115623794069E+01
                                 .691974464555551360E+01
 8
     .960511225893513772E+01
 9
                                 .346253542858316574E+00
                                -.972456776138063860E+01
10
     .690664382334616377E+01
                                -.127153702774917514E+02
    -.425518312218534476E+01
11
                                -.374581391519618043E+01
    -.132247394844809157E+02
15
                                 959749870725448356E+U1
    -.964940964293925416E+01
13
14
     .478446218423093526E+01
                                 .134650430069990693E+02
     .161490699506790776E+02
                                 .210043524055092697E+01
15
     .116971629307233777E+02
                                -.145143079484962967E+02
16
    -.600792278010507000E+01
                                -.192583713666365894E+02
17
    -.197670962617731023E+02
                                -.549919788496855717E+01
18
    -.144387563083162818E+02
                                 .143864375413314837E+02
19
     .653710035846073500E+01
                                 .200069013957453667E+02
50
     .226905695077822159E+02
21
                                 .385343224642388580E+01
                                -.193027848785530413E+02
     .164858798293849142E+02
22
23
    -.776048388102112145E+01
                                -.257995784541476185E+U2
    -.263080831078621465E+02
                                -.725197922730659346E+01
24
25
    -.192270960522314879E+02
                                 .191746242320445727E+02
     .828960479981791393E+01
                                 .265477020037199960E+02
26
     .292312252560682462E+02
27
                                 .56060A154746966378E+U1
     .212739758906446323E+02
                                -.240907773727864710E+U2
28
                               -.323401083121393248E+02
29
    -.951294630347011067E+01
                               -.900454199755674062E+01
    -.328485126180526948E+02
30
    -.240150255812371049E+02
                                 .239624804318400908E+U2
31
     .100420354930026864E+02
                                 .330880424432600672E+02
32
33
     .357714933211899700E+02
                                 .735858461507278359E+01
     .260617865845488105E+02
34
                                -.288785342526658239E+02
35
    -.112653525699867471E+02
                               -.388803110448877670E+02
    -.393886614382586794E+02
                               -.107570021766158347E+U2
36
    -.288027485126755739E+02
37
                                 .287501627061604107E+02
38
     .117944226084205117E+02
                                 .396281419230462033E+02
39
     .423115515641129800E+02
                                .911101296735373496E+01
     .3084944286315C1429E+02
                               -.336661590665168762E+02
40
    -.130177244196604406E+02
                               -.454203311141869394E+02
41
    -.459286492620155805E+02
                               -.125094062718317995E+02
42
    -.335903529813992478E+02
                                 .335377423262664600E+02
43
                                 .461680996068807707E+02
     .135467821076008393E+02
44
45
                                .108633981573127122E+02
     .488514835571688979E+02
     .356370062628139997E+02
                               -.384537025006850782E+02
46
47
    -.147700737937844664E+02
                               -.519602388946615882E+02
    -.524685362428206577E+02
                               -.142617764456253969E+02
48
49
    -.383778832760941133E+02
                                .383252563380191995E+02
     .152991230824665860E+02
                                .527079668439157774E+02
50
      PUV (COS THETA) =
                           .172632117610853773E+02
```

.934224388058987776E+02

```
THETA = 30.0 DEGREES
     MU = 1.00
                    NU = -.40
                                         QSP
              PSP
     .697966858638996821E+00
                                 .628452182304411485E+00
 1
     .264685191520384340E+01
                                -.477895826748892819E-02
 5
     .330621233937490457E+01
 3
                                -.310682586282114848E+01
     .113167472873922047E+00
                                -.598185637424308958E+01
 5
    -.504463119669334992E+01
                                -.430598599729680318E+01
                                 .210128348540353387E+01
 6
    -.640653836762939488E+01
                                 .723882918789294411E+01
 7
    -.700715814937718649E+00
                                 .451772209642955912E+01
     .767399068334997784E+01
 8
     .973929737145194587E+01
                                -.519878193372218645E+01
 9
     .151895254164084976E+01
                                -.126004136640069362E+02
10
    -.100740703602049736E+02
                                -.883361218431806506E+01
11
    -.128429842137007813E+02
                                 .419310330139032513E+01
12
    -.210759155632982071E+01
                                 .138592942672124732E+02
13
     .127042422373985747E+02
                                 .904663773030194484E+01
14
     .161768451448643877E+02
                                -.729067446642411825E+01
15
                                -.192216292444217389E+02
16
     .242617745455907570E+01
    -.151046975834117967E+02
                                -.133630428042707498E+02
17
    -.192810301328327556E+02
                                 .628505705493000789E+01
18
                                 .204808825004384981E+02
    -.351496873472032399E+01
19
     .177350733962494942E+02
                                 •135763252663073233E+02
50
                                -.938267072527891218E+01
     .226151586436168846E+02
21
     .433363378184129352E+01
                                -.258434296467531638E+02
25
                                -.178928768466724881E+02
23
    -. 201356516824096159E+02
    -.257195039759758281E+02
                                 .837708278051943836E+01
24
    -.492247118047054962E+01
25
                                 .271028152189872032E+02
     .227661072895875395E+02
                                 .181062507133749387E+02
26
     .290537361637662649E+02
                                -.114747174053124079E+02
27
     .574116534857742675E+01
                                -.324654504502882291E+02
28
                                -.224228631390790310E+02
    -.251667402897762156E+02
29
    -.321581524016316397E+02
                                 .104691447901283438E+02
30
                                 .337248976209089156E+02
31
    -.633002226599230782E+01
     .277972350408346169E+02
                                 .226362795435124783E+02
32
     .354924352219128022E+02
                                -.135667909268520052E+02
33
34
     .714873013361451158E+01
                                -.390875776187410508E+02
35
    -.301978970090111233E+02
                                -.269529228722355151E+02
    -.385968888768366575E+02
                                 .125612271584011747E+02
36
    -.773759702494758409E+01
                                 .403470583460094752E+02
37
                                 .271663624322998591E+02
     .328284137971896357E+02
38
     .419312001305265578E+02
                                -.156588802316660016E+02
39
                               -.457097641457587982E+02
40
     .855631237385581825E+01
    -.352290929189087573E+02
                                -.314830235585990435E+02
41
    -.450356758984390048E+02
                                 .146533219979978908E+02
42
    -.914518501734778537E+01
43
                                 .469692651391671698E+02
                                 .316964770925368949E+02
     .378596233205332080E+02
44
     .483700046889919775E+02
                                -.177509795556155915E+02
45
                                -.523319871474247411E+02
     .996390488212540711E+01
46
                                -. 360131493909243446E+02
47
    -. 402603134268898098E+02
    -.514744945989714737E+02
                                 .167454250043974343E+U2
48
    -.105527811346181843E+02
                                 .535915013072985069E+02
49
                                 .362266119971256837E+02
     .428908528201712561E+02
50
```

PUV (COS THETA) = .483971447812410585E+02 QUV (COS THETA) = .642099979395653848E+02

```
THETA = 30.0 DEGREES
                    NU =
     MU = 1.00
                          .05
              PSP
                                         OSP
 K
     .101062077464806388E+01
                                 .548722309768178576E+00
 1
     .295584166507395871E+01
                                -.643310584991070075E+00
 5
                                -. 405307177180721581E+01
     .286655392606633338E+01
 3
    -.112318773002931388E+01
                                -.621932474406924379E+01
 5
                                -.325746918279232646E+U1
    -.595649672473083955E+01
                                 .353746107080482009E+01
    -.577856521404204922E+01
 7
     .118677411960816040E+01
                                 .731933176192599338E+01
 8
     .890685255309004541E+01
                                 .258846161338620268E+01
     .864029785306437685E+01
 9
                                -.759085169897491206E+U1
    -.130023397830134701E+01
                                -.129881201142522409E+02
10
    -.119068297883696148E+02
                                -.648838984236070596E+01
11
                                 .707510489497352517E+01
    -.115516571789737009E+02
15
                                 .140877022724580950E+02
     .136394050635195950E+01
13
                                 .581916762746850694E+01
     .148569627168610664E+02
14
                                -.111284272705446441E+U2
15
     .144131743260753414E+02
    -.147743349253152553E+01
                                -.197563233563763508E+02
16
17
    -.178568391031640979E+02
                                -.971901068891708632E+01
    -.173244359986984570E+02
                                 .106126435955791169E+02
18
19
                                 .208558224213852506E+UZ
      .154115254310566210E+01
     .208069178479732331E+02
                                 .904974625844344581E+01
20
                                -.146659440668456133E+02
     .201859006411560886E+02
51
                                -.265243961808529518E+02
55
    -.165465121584454078E+01
                                -.129495653777860805E+02
23
    -.238067617643170768E+02
                                 .141501463788141362E+02
    -.230971308377111706E+02
24
25
     .171837318103011282E+01
                                 .276238657251902518E+02
26
     .267568195153772446E+02
                                 .122802860777029381E+02
27
     .259585751242462555E+02
                                -.182034373620818315E+02
28
    -.183187348192884735E+01
                                -.332924198283136574E+02
29
    -.297566490750111378E+02
                                -.161800953362420929E+U2
                                 .176876329587193282E+02
30
    -.288697913981527450E+02
                                 .343918756245714737E+02
     .189559641705165414E+01
31
     .327066965750283302E+02
32
                                 .155108091643049834E+02
33
      .317312257425396089E+02
                                -.217409190176262368E+U2
34
    -.200909732524161302E+01
                                -.400604197349202811E+U2
                               -.194106134440732626E+02
    -.357065185598430135E+02
35
    -.346424346693334104E+02
                                .212251108972733698E+02
36
37
     .207282065591027520E+01
                                 .411598680397949005E+02
                                 .187413235502324169E+02
38
     .386565603041746437E+02
39
     .375038634307509794E+02
                                -.252783940823158676E+U2
40
    -.218632183008823777E+01
                               -.468284063893940001E+02
41
    --416563778130589862E+02
                                -.226411249751022872E+U2
                                .247625836944945531E+02
42
    -.404150680157483369E+02
                                 .479278501690884677E+02
     .225004534436565807E+01
43
                                .219718328436869662E+02
     .446064160079661919E+02
44
     .432764933339999403E+02
                                -.288158650592646937E+02
45
    -.236354664806108261E+01
                                -.535963848992508685E+02
46
47
    -.476062306547070046E+02
                               -.258716324824123512E+02
48
    -.461876951424673313E+02
                                .283000531881377852E+02
49
     .242727025535972589E+01
                                 .546958257386133749E+02
     .505562665080598222E+02
                                 .252023389025749222E+02
50
```

```
THETA = 30.0 DEGREES
     MU = 1.00
                    NU = 1.50
             PSP
     .1605055266U8920260E+01
                                -.430073262355922606E+U0
                                -.323891997801202715E+01
     .235768347521206716E+01
 5
                                -.613047725282439587E+01
    -.533873799600301569E+00
 3
    -.560296435676092774E+01
                                -.477221853167297624E+U1
 4
 5
    -.725834217316422042E+01
                                 .140573558486653690E+01
    -.192816296228938916E+01
                                 .673591479574136816E+01
 6
     .645275581969027403E+01
 7
                                 .449025437627908886E+01
 8
     .899249081789166128E+01
                                -.498816567476933954E+01
                                -.127291361868518199E+02
 9
     .125152030580918091E+01
                                -.960238810182656566E+01
10
    -.104176624099737599E+02
                                 .316078049356102596E+01
11
    -.138375431279701317E+02
                                 .133044549198896054E+02
15
    -.369386870164155226E+01
     .112554985479275725E+02
                                 .929878403801128444E+01
13
                                -.674303697210327637E+01
     .155538915327123862E+02
14
     .301096673333675227E+01
                                -. 192859617714789103E+02
15
    -.152148567925160616E+02
                                -.144023670763344913E+02
16
                                 .491510733778554835E+U1
17
    -.203909584615884015E+02
    -.545057626577562264E+01
                                 .198554895335983272E+02
18
                                 .140945140828240971E+02
19
     .160496768176708196E+02
                               -.849691725861680738E+01
     .221030326014730651E+02
20
21
                                -.258337066926186971E+02
     .476624316747117537E+01
                                -.191956784740764117E+02
    -.200072154062045568E+02
22
    -. 269375431627792439E+02
                                 .666865682656515525E+01
23
                                 .264011855583600730E+02
24
    -.720501443098432621E+01
25
     .208408537451429487E+02
                                 .188863178295370454E+02
     .286479663300868718E+02
                                -.102502229978840680E+02
26
     .652014946177519881E+01
                                -.323780398661957410E+U2
27
                               -.239864790830609415E+02
28
    -.247975817356863500E+02
                                 .842178025431221480E+01
29
    -.334813486532338791E+02
    -.895856268980455312E+01
                                 .329445662177415408E+02
30
     .256306399637202348E+02
31
                                 .236764172998931273E+02
     .351909665967207611E+02
                                -.120032074314187324E+02
32
     .827344546369540946E+01
                                -.389207265644440841E+02
33
    -.295869385359049418E+02
                               -.287760692466189704E+02
34
    -. 400237540963389895E+02
                                 .101746566941463420E+02
35
    -.107116744099945910E+02
                                 .394867363804907405E+02
36
     .304196700153264430E+02
                                 .284656258581196086E+02
37
     .417329201617313212E+02
38
                                -.137559984869994362E+02
     .100264185371621999E+02
                               -.454625001115685575E+02
39
40
    -.343757142003168053E+02
                               -.335649845023415423E+02
                                 .119273791728880635E+02
    -.465653563088775195E+02
41
                                 .460281948894104779E+02
    -.124645405923551052E+02
42
     .352082437590199829E+02
                                 .332543108215164591E+02
43
     .482742437804046013E+02
                               -.155086651099869740E+U2
44
     .117792005975916777E+02
                               -.520037082927998976E+02
45
                               -.383534853414406587E+02
    -.391641249915237934E+02
46
    -.531064552954426899E+02
                                 .136799997286918894E+U2
47
    -.142172552763111882E+02
                                 .525691997478233910E+02
49
                                 .380426621368678473E+02
49
     .399965211458351181E+02
                                -.172612472445473074E+02
50
     .548151590028692997E+02
```

PUV (COS THETA) = .618522834598657480E+02 QUV (COS THETA) = -.305947641500301026E+02

```
30.0 DEGREES
     MU = 1.50
                   NU =-1.50
                                  THETA =
             PSP
                                        QSP
     .4999999999999701-100
                                .866025403784438128-100
 1
 2
     .1999999999999744E+01
                                .151461293802434073E-27
                               -.519615242270662523E+01
      4999999999999360E+01
 3
    -.9999999999998720E+00
                                -.155884572681198757E+02
    -.2099999999999731E+02
 5
                                -.155884572681198757E+02
    -.35999999999999539E+02
                                .103923048454132505E+02
 6
 7
    -.14999999999999808E+02
                                .467653718043596271E+U2
     .4099999999999475E+02
 8
                                .467653718043596271E+U2
 9
      .76999999999999022E+02
                               -.155884572681198771E+02
     .31999999999999598E+02
10
                               -.935307436087192555E+U2
      77999999999998962E+02
11
                               -.935307436087192555E+02
    -.14399999999999811E+03
                                .207846096908264986E+02
12
    -.6599999999999113E+02
                                .155884572681198754E+03
13
     .11599999999999855E+03
                                .155884572681198754E+03
14
     .22099999999999721E+03
                               -.259807621135331287E+02
15
     .10099999999999875E+03
                               -.233826859021798138E+03
16
17
    -.17099999999999777E+03
                               -.233826859021798138E+03
    -.32399999999999581E+03
                                .311769145362397488E+02
18
    -.15299999999999800E+03
19
                                .327357602630517387E+03
     .22699999999999713E+03
                                .327357602630517387E+03
20
     .43699999999999444E+03
21
                               -.363730669589463791E+U2
     .20599999999999740E+03
                               -.436476803507356521E+03
22
                               -. 436476803507356521E+03
    -.2999999999999612E+03
23
                                .415692193816529993E+02
    -.575999999999999258E+03
24
    -.27599999999999643E+03
                                .561184461652315522E+03
25
     .37399999999999525E+03
                                .561184461652315522E+03
26
      .724999999999999076E+03
                               -.467653718043596296E+02
27
28
     .34699999999999560E+03
                               -.701480577065394408E+03
    -.46499999999999401E+03
29
                               -.701480577065394408E+03
    -. 89999999999998844E+03
                                .519615242270662498E+02
30
    -. 43499999999999439E+03
                                .857365149746593160E+03
31
     .556999999999999291E+03
                                .857365149746593160E+03
32
     .10849999999999861E+04
                               -.571576766497728800E+02
33
     .52399999999999333E+03
                               -.102883817969591180E+04
34
    -.66599999999999143E+n3
                               -.102883817969591180E+04
35
36
    -.12959999999999834E+04
                                .623538290724795002E+02
37
    -.62999999999999189E+03
                                .121589966691335030E+04
38
     .77599999999999010E+03
                                .121589966691335030E+04
39
     .151699999999999806E+04
                               -.675499814951861305E+02
     .73699999999999060E+03
40
                               -.141854961139890869E+04
    -.90299999999998840E+03
41
                               -.141854961139890869E+04
    -- 17639999999999774E+04
                                .727461339178927507E+02
42
    -_86099999999998894E+03
                                .163678801315258694E+04
43
     .10309999999999868E+04
                                .163678801315258694E+04
44
                               -.779422863405993810E+02
     .20209999999999742E+04
45
     .98599999999998742E+03
                               -.187061487217438508E+04
46
    -.11759999999999849E+04
                               -.187061487217438508E+04
47
    -.23039999999999705E+04
48
                                .831384387633060011E+02
    -.11279999999999855E+04
49
                                .212003018846430309E+04
     .132199999999999831E+04
                                .212003018846430309E+04
50
      PUV (COS THETA) =
                           .149171725890026572E+04
      QUV (COS THETA) =
                           .375765567157950091E+04
```

```
THETA = 30.0 DEGREES
                    NU = -.40
     MU = 1.50
                                         OSP
             PSP
     .980107176607203394E+00
                                 .499389549713500831E+00
 1
 5
     .450252356197797981E+01
                                -.178809439734960999E+01
     .401579916891860279E+01
 3
                                -.1107534907U5671351E+02
    -.105967078277706114E+02
                                -.185207932662956930E+U2
 5
                                -.463249787341252016E+01
    -.319828073103788968E+02
    -.300673113118871550E+02
                                 .319173430986048368E+02
 6
 7
     .142157129402746711E+02
                                 .544806709356602835E+02
 8
     .685615657431380786E+02
                                 .191880614666865737E+02
     .642752509268409841E+02
 9
                               -.625996974297129237E+U2
                               -.108452737903407089E+03
    -.257164080161840510E+02
10
    -.128118084362320193E+03
                               -.419523117280723681E+02
11
    -.120518903515844758E+03
                                 .103048696718291572E+03
12
13
     .312195075534340943E+02
                                 .180363278823936292E+03
14
     .196773077665860586E+03
                                 .728515333119700833E+02
     .184918983576833823E+03
15
                                -.153338056309940609E+03
    -.446042970540894446E+02
                               -.270286009042847713E+03
16
17
    -.288405831155823898E+03
                               -.111959441563979543E+03
                                 .213394060859060208E+03
    -.271354776611872819E+03
18
19
     .519914910160854642E+02
                                 .378147213214541528E+03
     .389137059330145492E+03
                                 .159202321138500921E+03
50
                               -.283290425711250195E+03
     .365946997118897110E+03
21
    -.672603749414867901E+02
                               -.504020606684617563E+03
22
    -.512846047690890006E+03
                               -.214653887381134045E+03
23
    -.482574930599971332E+03
                                 .362953435520910744E+03
24
25
     .765316633282287853E+02
                                 .647832474107475991E+03
     .645653510735992803E+03
                                 .278240424946279087E+03
26
     .607359291553030850E+03
27
                               -. 452456805633641681E+03
85
    -.936846416783760867E+02
                               -.809656530828716639E+03
29
    -.801438733967518519E+03
                               -.350035649179535873E+03
30
    -.754179365480140299E+03
                                 .551726820703843180E+03
31
     .104840024489864058E+03
                                 .989419061502739681E+03
32
     .966322431883402518E+03
                                .429965844735304579E+03
     .909155866879235042E+03
                               -.660837196077115067E+03
33
34
    -.123877097264757335E+03
                               -.118719378147514494E+04
35
    -- 115418388998570944E+04
                                -.518104726959185029E+03
    -.108616808125237972E+04
                                .779714216407857515E+03
36
     .136916574500991281E+03
                                .140290697540033260E+04
37
     .135114382277237464E+04
                                .614378580505577398E+03
38
     .127133672309750969E+04
39
                               -.908431597041670352E+03
    -.157837741700630533E+03
40
                               -.163663235862390247E+04
    -.157108151574546276E+04
                               -.718861120720081512E+03
41
42
    -.147854107791668959E+04
                                .104691562263295375E+04
43
     .172761313361610455E+03
                                 .188829621580025474E+04
     .180011768340290916E+04
                                 .831478632257097545E+03
44
45
     .169390186020785479E+04
                               -.119524000852730754E+04
                               -.215797226227498923E+04
    -.195566574985995683E+03
46
47
    -.205213161124677848E+04
                               -.952304830462225322E+03
    -.193129835547306991E+04
                                .135333103937913189E+04
48
     .212374241071721581E+03
                                .244558678270250611E+04
49
     .231324401377500609E+04
                                .108126599998986502E+04
50
      PUV (COS THETA) =
                           .261021635355212178E+04
```

.191649408553523991E+04

QUV (COS THETA) =

```
MU = 1.50
                                   THETA =
                                           30.0 DEGREES
                    NU =
                          . 05
             PSP
                                         OSP
     .150717337661639680E+01
                                 .361840313976652925E+00
 1
                                -.333756897818600929E+U1
 2
     .501778171265453763E+01
                                -.135489991545322522E+U2
     .199301829155776563E+01
 3
    -.157041142596799259E+02
                                -.177977047767097253E+02
 4
                                 .233143401888123093E+01
 5
    -.348059537351815745E+02
                                 .400130495992011695E+02
 6
    -.236441506883174299E+02
                                 .523506370789857548E+02
 7
     .277455996046997127E+02
                                 .273503010174534384E+01
 9
     .748290525821524250E+02
                                -.796755261101757451E+02
 9
     .504179337048503067E+02
                                -.104304011996973734E+03
10
    -.521670928971044401E+02
    -.139622541738995771E+03
                                -. 121451981598627082E+02
11
    -.968498308265850797E+02
                                 .132253053911286980E+03
15
                                 .173374454754504665E+03
     .744331306514654340E+02
13
                                 .256156953793018628E+02
     .214650957720282941E+03
14
     .148404378568093075E+03
                                -.198029007778703879E+03
15
    -.109079176353211359E+03
                                -.259845340127747553E+03
16
17
    -.314449764011442597E+03
                                -.434298965362318138E+02
    -.219617040414802957E+03
                                 .276720012936257435E+03
18
19
     .141569766516913554E+03
                                 .363433293340533390E+03
                                 .653044258544835522E+02
20
     .424483497127046080E+03
                                -.368609444160116657E+03
     .295952352881286064E+03
21
                                -.484421689169031183E+03
    -.186440364628n00681E+03
55
                                -.915226611102260872E+02
    -.559287620552522049E+03
23
                                 .473413926674112534E+03
    -.391945779452971057E+03
24
25
     .229155507201044077E+03
                                 .622527152837071925E+03
26
     .704326670802441844E+03
                                 .121801224527290410E+03
     .493061856644429276E+03
                                -.591416835254414077E+03
27
28
    -.284250657721472404E+03
                                -.778033059120824624E+03
29
                                -.156423491881845529E+03
    -.874136111362234126E+03
    -.613836047941089380E+03
                                 .722334795124852276E+03
30
                                 .950656033244120272E+03
     .337190352703857002E+03
31
                                 .195106088397722435E+03
     .105418047874647023E+04
32
     .739732889857522711E+03
                               -.866451181061596140E+03
33
34
    -.402510055633626531E+03
                               -.114067944998312788E+04
35
    -. 125899523644057883E+04
                               -.238132388851090139E+03
    -.885287845879157927E+03
                                 .102348261828847666E+04
36
     .465674303025352329E+03
37
                                 .134781993456167843E+04
                                 .285219018465779629E+03
38
     -147404492095913125E+04
39
     .103596545252056637E+04
                                -.119371248158166285E+04
                               -.157236086175594094E+04
    -.541218558364463059E+03
40
                                -.336649352017959916E+03
    -.171386499578755616E+04
41
    -.120630117326717670E+04
                                 .137685739616498569E+04
42
     .514607358165530059E+03
                                 .181401865678974640E+04
43
44
     -196391999744042489E+04
                                 .392140014731461991E+03
45
     .138175954463356025E+04
                                -.157320073681461420E+04
    -.700376165913981990E+03
                               -.207307729443926381E+04
46
47
    -.223874538940316611E+04
                               --451974381382454862E+03
    -.157687603010514569E+04
                                 .178245912875437936E+04
48
     .783989518124390191E+03
                                 .234925279992832418E+04
49
                                 .515869077194769521E+03
50
     .252380570819035116E+04
                           .284780978291872870E+04
      PUV (COS THETA) =
```

.914354132436944155E+03

QUV (COS THETA) =

```
30.0 DEGREES
     MU = 1.50
                   NU = 1.50
                                  THETA =
                                        QSP
             PSP
     .259807621135331261E+01
                               -.14999999999999808E+U1
                               -.94999999999998784E+01
     .259807621135331261E+01
                               -.1699999999999782E+02
 3
    -.103923048454132505E+02
    -.311769145362397514E+02
                               -.49999999999999360E+U1
 5
    -.311769145362397514E+02
                                .2999999999999616E+02
     .103923048454132505E+02
 6
                                .53999999999999309E+02
     .649519052838328166E+02
                                .22499999999999705E+02
 7
     .649519052838328166E+02
 8
                               -.57499999999999253E+02
                               -.10699999999999863E+03
 9
    -.207846096908265011E+02
                               -.46999999999999410E+02
    -.124707658144959003E+03
10
                                .95999999999998749E+02
    -.124707658144959003E+03
11
     .207846096908265029E+02
                                •17999999999999767E+03
12
     .189659563428791822E+03
                                .824999999999998921E+02
13
     .189659563428791822E+03
                               -.141499999999999821E+03
14
15
    -.311769145362397495E+02
                               -.26899999999999657E+03
                               -.12499999999999842E+03
16
     -.280592230826157760E+03
                                .1979999999999744E+03
    -.280592230826157760E+03
17
     .311769145362397533E+02
                                .37799999999999514E+03
18
     .376721050646230331E+03
                                .17849999999999769E+03
19
                               -.26149999999999667E+03
     .376721050646230331E+03
50
                               -.50299999999999358E+03
    -.415692193816529999E+02
21
22
    -.498830632579836020E+03
                               -.23899999999999696E+03
    -.498830632579836020E+03
                                .33599999999999568E+03
23
24
     .415692193816530037E+02
                                .64799999999999168E+03
                                .31049999999999600E+03
25
     .626136366936148342E+03
                               -.417499999999999468E+03
26
     .626136366936148342E+03
                               -.808999999999998966E+03
27
    -.519615242270662504E+02
                               -.38899999999999504E+03
28
    -.779422863405993782E+03
    -.779422863405993782E+03
29
                                .50999999999999345E+03
     .519615242270662542E+02
                                .98999999999998730E+03
30
     .937905512298545855E+03
31
                                .47849999999999385E+03
     .937905512298545855E+03
                               -.60949999999999222E+03
32
33
    -.623538290724795009E+02
                               -.118699999999999848E+04
     -.112236892330463105E+04
                               -.57499999999999266E+03
34
    -.112236892330463105E+04
                                .71999999999999076E+03
35
     .623538290724795046E+02
                                .14039999999999820E+04
36
                                .68249999999999124E+03
37
     .131202848673342287E+04
     .131202848673342287E+04
                               -.8374999999999998930E+03
38
    -.727461339178927513E+02
                               -.16369999999999791E+04
39
    -.152766881227574782E+04
                               -.79699999999998982E+03
40
    -.152766881227574782E+04
                                .96599999999998761E+03
41
     .727461339178927551E+02
                                .18899999999999758E+04
42
     .174850529024077939E+04
                                .92249999999998817E+03
43
     .174850529024077939E+04
                               -.110149999999999859E+04
44
45
    -.831384387633060018E+02
                               -.21589999999999724E+04
46
    -.199532253031934409E+04
                               -.10549999999999865E+04
47
    -.199532253031934409E+04
                                .12479999999999840E+04
     .831384387633060055E+02
                                .24479999999999686E+04
48
                                .11984999999999846E+04
49
     .224733592282061541E+04
                               -.14014999999999821E+04
50
     .224733592282061541E+04
```

CONVERGENCE OF THE HYPERGEOMETRIC SERIES ON |z|=1

We consider the series

$$\sum_{n=0}^{\infty} d_n z^n,$$

where

$$d_{n} = \frac{(a)_{n}(b)_{n}}{n!(c)_{n}}$$

and

$$|Z|=1$$
.

By an extension of a Weierstrass test for convergence (see, e.g., reference 7, page 401), if

$$\frac{d_{n+1}}{d_n} = 1 - \frac{\alpha}{n} - \frac{A_n}{n^{\lambda}},$$

where α is complex and arbitrary, $\lambda\!>\!1$, and where the \boldsymbol{A}_n are bounded, then the series

$$\sum_{n=0}^{\infty} d_n Z^n, |Z|=1,$$

- I) converges absolutely for $Re\{\alpha\}>1$,
- II) converges conditionally if Z#1 and $0<Re\{\alpha\}\le 1$,
- III) diverges for $Re\{\alpha\} \le 0$.

To use this test in the present case, choose $\alpha=c+1-a-b$, $\lambda=2$, and

$$A_{n} = -\alpha \left[\frac{c+1+\frac{c}{n} + \frac{ab-c}{\alpha}}{1+\frac{1}{n}(c+1) + \frac{c}{n^{2}}} \right].$$

The An are clearly bounded, and one has

$$n \begin{bmatrix} \frac{d}{d_{n+1}} - 1 \end{bmatrix} = n \begin{bmatrix} \frac{(a)_{n+1}(b)_{n+1}}{(n+1)!(c)_{n+1}} \\ \frac{(a)_{n}(b)_{n}}{n!(c)_{n}} - 1 \end{bmatrix}$$

$$= \frac{\frac{ab-c}{n} - \alpha}{\frac{c}{n^2} + \frac{(c+1)}{n} + 1}$$

Thus

$$\frac{d_{n+1}}{d_n} = 1 - \frac{1}{n} \left(\frac{\alpha - \frac{ab-c}{n}}{1 + \frac{1}{n}(c+1) + \frac{c}{n^2}} \right)$$

$$= 1 - \frac{\alpha}{n} + \left[\frac{\alpha}{n} - \frac{\frac{\alpha}{n} - \frac{ab - c}{2}}{1 + \frac{1}{n}(c + 1) + \frac{c}{n^2}} \right]$$

$$= 1 - \frac{\alpha}{n} + \left[\frac{\frac{\alpha(c+1)}{n^2} + \frac{\alpha c}{n^3} + \frac{ab-c}{n^2}}{1 + \frac{1}{n}(c+1) + \frac{c}{n^2}} \right]$$

$$= 1 - \frac{\alpha}{n} - \frac{1}{n^2} \left(-\alpha\right) \left[\frac{c+1 + \frac{c}{n} + \frac{ab-c}{\alpha}}{1 + \frac{1}{n} (c+1) + \frac{c}{n^2}} \right]$$

$$= 1 - \frac{\alpha}{n} - \frac{A_n}{n^2},$$

as required.

Then since Re $\{\alpha\}$ = Re $\{c+1-a-b\}$ = - 2 Re $\{\mu\}$ +1, we have the desired results.

APPENDIX C

AN ORDER RELATION FOR LOG $\Gamma(z+a)$ FOR LARGE Z

From Stirling's formula (see, e.g. references 2, 5, and 8), we have

$$\log \Gamma(z) = (z-\frac{1}{2}) \log z - z+\frac{1}{2}\log(2\pi)+O(\frac{1}{z}).$$

Then

$$\log \Gamma(z+a) = (z+a-\frac{1}{2})\log(z+a) - (z+a) + \frac{1}{2}\log(2\pi) + O(\frac{1}{z+a})$$

$$= (z+a-\frac{1}{2})\log(z+a) - (z+a) + \frac{1}{2}\log(2\pi) + O(\frac{1}{z}),$$
since $\frac{1}{z+a} = \frac{1}{z} - \frac{a^2}{z^2} + \frac{a^2}{z^3} - \dots$

It follows that

log
$$\Gamma(z+a)$$
-log $\Gamma(z) = (z+a-\frac{1}{z})\log(z+a) - (z+a)$
$$-(z-\frac{1}{z})\log z + z + O(\frac{1}{z}).$$

Writing $z+a = z(1+\frac{a}{z})$, we have

log
$$\Gamma(z+a)$$
-log $\Gamma(z) = (z+a-\frac{1}{2}) (\log z + \log(1+\frac{a}{z})) - z - a$

$$-(z-\frac{1}{2}) \log z + z + O(\frac{1}{z})$$

$$= a \log z + (z+a-\frac{1}{2}) \log(1+\frac{a}{z}) - a + O(\frac{1}{z}).$$

Now if |z| is large enough so that $|\frac{a}{z}| < 1$,

$$\log(1+\frac{a}{z}) = \frac{a}{z} - \frac{1}{2}(\frac{a}{z})^2 + \dots$$
, and then
$$(z+a-\frac{1}{2})\log(1+\frac{a}{z}) = a + O(\frac{1}{z}).$$

It follows that when |z| > |a|,

$$\log \Gamma(z+a) = \log \Gamma(z) + a \log z + O(\frac{1}{z}).$$

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